ANALYSIS AND TOPOLOGY—EXAMPLES 4

(updated 24 November 2024)

Exercises

1. Recall from lectures that a non-empty topological space (X, τ) is said to be connected if there is no separation of X by open, non-empty, disjoint subsets. Show that (X, τ) is connected if and only if the only subsets of X that are open and closed in X are $\emptyset$ and X itself.

2. Let $(X, \|\cdot\|)$ be a normed vector space over $\mathbb{R}$ (as defined in Linear Algebra), and recall that the norm induces a canonical metric on X given by $d(x, y) = \|x - y\|$. Show that the open and closed balls of such (X, d) are path connected.
If you are not yet familiar with the concept of norm: it's a function $\|\cdot\| \colon X \to \mathbb{R}$ which is positive definite ($\|x\| \geq 0$ with equality iff $x = 0$), subadditive (obeys a triangle inequality $\|x - y\| \leq \|x\| + \|y\|$) and absolutely homogeneous with respect to scalar multiplication ($\|\lambda x\| = |\lambda|\|x\|$).

3. Show that if (X, τ) is path-connected, then it is also connected.

4. Show that the relation $\sim$ on X defined by $x \sim y$ if and only if $x, y \in X$ are path-connected is an equivalence relation. (Recall from lectures that the equivalence classes are called the path-components of X.) *Hint: apply question 8 from sheet 3.*

5. Verify that the following properties of differentiation in single-variable remain true in multiple variables.
- (a) (Sum and scalar multiplication) Let functions $f, g \colon \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $x_0 \in \mathbb{R}^n$. Show that $\lambda f + g$, where $\lambda \in \mathbb{R}$, is also differentiable at $x_0 \in \mathbb{R}^n$. What is its derivative?
- (b) (Product rule) Let $f \colon \mathbb{R}^n \to \mathbb{R}^m$ and $g \colon \mathbb{R}^n \to \mathbb{R}^d$ be differentiable at $x_0 \in \mathbb{R}^n$. Show that $f \cdot g$, where $\cdot$ denotes the inner product, is also differentiable at $x_0 \in \mathbb{R}^n$, with $D(fg)|_{x_0} = Df|_{x_0} \cdot g(x_0) + f(x_0) \cdot Dg|_{x_0}$.
- (c) (Chain rule) Let $f \colon \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $x_0 \in \mathbb{R}^n$ and $g \colon \mathbb{R}^m \to \mathbb{R}^d$ be differentiable at $f(x_0) \in \mathbb{R}^m$. Show that $f \circ g$ is differentiable at $x_0 \in \mathbb{R}^n$, with $D(f \circ g)|_{x_0}(h) = Df|_{g(x_0)}(Dg|_{x_0}(h))$.

Problems

6. Which of the following subsets of $\mathbb{R}^2$ with the Euclidean topology are connected? Which are path-connected? (And why?)
- (a) $\{(x, y) \in \mathbb{R}^2 : \|(x, y) - (-1, 0)\| \leqslant 1 \text{ or } \|(x, y) - (1, 0)\| < 1\}$;
- (b) $\{(x, y) \in \mathbb{R}^2 \colon x = 0 \text{ or } y = qx \text{ for } q \in \mathbb{Q}\}$;
- (c) $\{(x, y) \in \mathbb{R}^2 \colon x = 0 \text{ or } y = qx \text{ for } q \in \mathbb{Q}\} \backslash \{(0, 0)\}$;
- (d) $S \subset \mathbb{R}^2$ any star-shaped domain, i.e. a set with the property that there is a point $x_0 \in S$ such that for all $x \in S$, the line segment between x_0 and x is contained in S.

7. Let $K_1 \supset K_2 \supset K_3 \supset \cdots$ be a decreasing sequence of non-empty, connected, compact subsets of a Hausdorff space X. Let $K = \bigcap_{n=1}^{\infty} K_n$.
- (a) Show that K is non-empty.
- (b) Show that K is connected.
- (c) Give an example with $X = \mathbb{R}^2$ to show that the conclusion of part (b) need not be true if K_n are assumed to be "closed" instead of "compact".

8. Show that if $f\colon \mathbb{S}^1 \to \mathbb{R}$ is continuous then there is an $x \in \mathbb{S}^1$ such that $f(-x) = f(x)$. Deduce that at any point in time there are antipodal locations on the Earth's equator that have the same temperature.

9. ($\star$) Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be a function under which the image of each path-connected set is path-connected and the image of each compact set is compact. Show that f is continuous.

10. ($\star$) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $(x_0, y_0) \in \mathbb{R}^2$.

(a) Suppose that $\partial_1 f$ exists and is continuous in some open ball around (x_0, y_0), and that $\partial_2 f$ exists at (x_0, y_0). Show that f is differentiable at (x_0, y_0).

(b) Suppose instead that $\partial_1 f$ exists and is bounded on some open ball around (x_0, y_0), and that for fixed x the function $y \mapsto f(x, y)$ is continuous. Show that f is continuous at (x_0, y_0).

11. Consider the map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $f(x) = x/\|x\|$ for $x \neq 0$, and $f(0) = 0$.

(a) Using the definition of derivative, show that $x \mapsto \|x\|^2$ is differentiable and compute its derivative. *Note: please avoid using partial derivatives.*

(b) Using part (a) and questions 5(b) and 5(c), show that f is differentiable except at 0, and that

$$Df|_x(h) = \frac{h}{\|x\|} - \frac{x(x \cdot h)}{\|x\|^3}.$$

Note: please avoid using partial derivatives.

(c) Verify that $Df|_x(h)$ is orthogonal to x and explain geometrically why that is.

12. Define $f\colon \mathcal{M}_n \to \mathcal{M}_n$ by $f(A) = A^2$.

(a) Show that f is continuously differentiable on the whole of $\mathcal{M}_n$.

(b) Deduce that there is a continuous square root function on some neighbourhood of the identity Id; that is, show that there is an open ball $B_\varepsilon(\mathrm{Id})$ for some $\varepsilon > 0$ and a continuous function $g\colon B_\varepsilon(\mathrm{Id}) \to \mathcal{M}_n$ such that $g(A)^2 = A$ for all $A \in B_\varepsilon(\mathrm{Id})$.

(c) Is it possible to define a continuous square-root function on the whole of $\mathcal{M}_n$?

13. Consider the function $\det\colon \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}$, where $\mathrm{GL}_n(\mathbb{R}) = \{X \in \mathcal{M}_n \colon X \text{ invertible}\}$.

(a) Show that $\mathrm{GL}_n(\mathbb{R})$ is an open subset of $\mathcal{M}_n$.

(b) Show that det is differentiable over all of $\mathrm{GL}_n(\mathbb{R})$, with $D\det|_A(H) = \det A \operatorname{tr}(A^{-1}H)$.

(c) Show that det is twice differentiable at I and find $D^2\det|_I$ as a bilinear map.

14. Let $f_n\colon I \to \mathbb{R}$, with $I \subset \mathbb{R}$, be given by:

(a) $f_n(x) = xe^{-nx}$ and $I = [0, +\infty)$;

(b) $f_n(x) = \dfrac{n^2 x}{1 + n^4 x^2}$ on $I = [0, 1]$;

(c) $f_n(x) = (-1)^n \dfrac{x^n}{n}$ on $I = [0, 1]$.

For each case, determine if the series $\sum_{n=1}^{\infty} f_n$ converges or diverges in I. In the former case, justify if the convergence is pointwise, uniform, pointwise absolute, uniform absolute.

15. Consider the series of functions $\sum_{n=1}^{\infty} (x - n)^{-2}$, for $x \in X := \mathbb{R}\backslash\mathbb{N} \to \mathbb{R}$.

(a) Show that the series converges pointwise on X.

(b) Does the series converge uniformly on X?

(c) Show that the series is continuous, i.e. $f(x) = \sum_{n=1}^{\infty} (x - n)^{-2} \in C(X)$.

OPTIONAL extra problems (not for marking)

16. Let $f\colon U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^{n+m}$ open, be a continuously differentiable function. (By convention, we use the notation $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$.) Let $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}^m$ be such that $(x_0, y_0) \in U$. Suppose $f(x_0, y_0) = 0$ and that

$$J_y f|_{(x_0,y_0)} := \det\left(\left[\frac{\partial f^j}{\partial y^i}\right]_{i,j=0}^{m}\right) \neq 0$$

(a) Let $F(x,y) = (x, f(x,y))$. Show that F is locally invertible around the point $(x_0, 0)$.

(b) Let $G = (G^1, \ldots, G^{n+m})$ denote the local inverse of f. Show that, for points $(X, 0)$, $X \in \mathbb{R}^n$, in the domain of G, one has $f(X, G^n(X,0), \ldots G^{n+m}(X,0)) = 0$.

(c) Deduce that there exists an open neighborhood $W' \subset \mathbb{R}^n$ of x_0 and an open neighborhood $W'' \subset \mathbb{R}^m$ of y_0 satisfying $W' \times W'' \subset U$, and a unique map $g \colon W' \to W''$ such that

$$\begin{cases} g(x_0) = y_0 \\ f(x, g(x)) = 0 \end{cases}.$$

We call g the *implicit* function defined by the zero set of f.

(d) Show that g is differentiable and compute its derivative.

(e) As an application, show that the level set $C = \{(x,y) \in \mathbb{R}^2 \colon x^3 + y^3 - 3yx = 0\}$ is locally graphical, except in small neighborhoods of $(0,0)$ and $(2^{\frac{2}{3}}, 2^{\frac{1}{3}})$.

17. Let (X, τ) be a compact Hausdorff space. Let $K \subset X$ be closed and $f \colon K \to [-1,1]$ be a continuous function (with respect to subspace topology on K).

(a) Show that there exists a continuous $g_1 \colon X \to \mathbb{R}$ such that $\sup_X |g_1| \leq \frac{1}{3}$ and $\sup_K |f - g_1| \leq \frac{2}{3}$. *You may assume the following statement: for any closed disjoint sets $A, B \subset X$ and $-\infty < a < b < \infty$, one can find a continuous bump function $\chi \colon X \to [a,b]$ such that $\chi(x) = a$ for $x \in A$ and $\chi(x) = b$ for $x \in B$.*

(b) For $n \geq 1$, construct continuous functions $g_n \colon X \to \mathbb{R}$ such that

$$\sup_X |g_n| \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} \quad \text{and} \quad \sup_K |f - \sum_{j=1}^{n} g_j| \leq \left(\frac{2}{3}\right)^n.$$

(c) Deduce that the series $g(x) = \sum_{j=0}^{\infty} g_n(x)$ is a well-defined continuous real-valued function on X such that $g|_K = f$.